

# Commutator estimates in Besov-Morrey spaces with applications to the well-posedness of the Euler equations and ideal MHD system

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## Abstract

We develop commutator estimates in the framework of Besov-Morrey spaces, which are modeled on Besov spaces and the underlying norm is of Morrey space rather than the usual  $L^p$  space. As direct applications of commutator estimates, we establish the local well-posedness and blow-up criterion of solutions in Besov-Morrey spaces for the incompressible Euler equations and ideal MHD system. Main analysis tools are the Littlewood-Paley decomposition and Bony's para-product formula.

**Keywords.** Well-posedness; Euler equations; ideal MHD system; Besov-Morrey spaces

**AMS subject classification:** 35L25; 35L45; 76N15

## 1 Introduction

### 1.1 Euler equations and MHD system

In this paper, the one interest is to consider the incompressible Euler equations for perfect fluid

$$\begin{cases} \partial_t v + (v \cdot \nabla)v + \nabla P = 0, & (x, t) \in \mathbb{R}^n \times (0, \infty), \\ \operatorname{div} v = 0, & (x, t) \in \mathbb{R}^n \times (0, \infty), \\ v(x, 0) = v_0(x), & x \in \mathbb{R}^n, \end{cases} \quad (1.1)$$

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where  $n \geq 2$ ,  $v = v(x, t) = (v^1, v^2, \dots, v^n)$  stands for the velocity of the fluid,  $P = P(x, t)$  is the pressure, and  $v_0(x)$  is the given initial velocity satisfying  $\operatorname{div} v_0 = 0$ .

The other interest is to consider the ideal magneto-hydrodynamics (MHD) system

$$\begin{cases} \partial_t v + (v \cdot \nabla)v - (b \cdot \nabla)b + \nabla \Pi = 0, & (x, t) \in \mathbb{R}^n \times (0, \infty), \\ \partial_t b + (v \cdot \nabla)b - (b \cdot \nabla)v = 0, & (x, t) \in \mathbb{R}^n \times (0, \infty), \\ \operatorname{div} v = 0, \quad \operatorname{div} b = 0, & (x, t) \in \mathbb{R}^n \times (0, \infty), \\ v(x, 0) = v_0(x), \quad b(x, 0) = b_0(x), & x \in \mathbb{R}^n, \end{cases} \quad (1.2)$$

where  $n \geq 2$ ,  $v = v(x, t) = (v^1, v^2, \dots, v^n)$  denotes the velocity of the fluid,  $b = b(x, t) = (b^1, b^2, \dots, b^n)$  denotes the magnetic field, and  $\Pi = P(x, t) + \frac{1}{2}|b(x, t)|^2$  is the total pressure.  $v_0(x)$  and  $b_0$  are the initial velocity and initial magnetic fields satisfying  $\operatorname{div} v_0 = 0$  and  $\operatorname{div} b_0 = 0$ , respectively.

## 1.2 Related results

For the well-posedness of the system (1.1), there are many results available. Given  $v_0 \in H^s$ ,  $s > 1 + n/2$ , Kato [9] established the local existence and uniqueness of regular solution belonging to  $C([0, T]; H^s(\mathbb{R}^n))$  with  $T = T(\|v_0\|_{H^s(\mathbb{R}^n)})$ . Later, various function spaces are used to consider the well-posedness for the incompressible Euler equations. Kato and Ponce [10] extended the result to the fractional-order Sobolev space  $W^{s,p}$  with  $s > 1 + n/p$ ,  $1 < p < \infty$ . Vishik [21] showed the global well-posedness in the critical Besov space  $B_{p,1}^{1+2/p}(\mathbb{R}^2)$  ( $1 < p < \infty$ ). Subsequently, Vishik [22] proved the existence ( $n = 2$ ) and uniqueness ( $n \geq 2$ ) result for (1.1) with initial vorticity belonging to a space of Besov type. In [23], the second author generalized the results of Vishik in critical Besov space  $B_{p,1}^{1+n/p}$  ( $1 < p < \infty$ ,  $n \geq 3$ ). Pak and Park [17] considered the endpoint Besov space  $B_{\infty,1}^1(\mathbb{R}^n)$  and proved the corresponding results. Chae [4] studied the case of the initial data belonging to the Triebel-Lizorkin space. Based on [4], Chen, Miao and Zhang [6] studied the local well-posedness of the ideal MHD system (1.2) in the Triebel-Lizorkin space. Miao and Yuan [16] established the existence results for (1.2) in the critical Besov space  $B_{p,1}^{1+n/p}(\mathbb{R}^n)$  ( $1 \leq p \leq \infty$ ).

For the blow-up criterion of solutions, Beale, Kato and Majda [2] showed a celebrated criterion for solutions in terms of the vorticity  $\omega = \nabla \times v$ , namely,  $\limsup_{t \rightarrow T_*} \|v(t)\|_{H^s} = \infty$  if only if  $\int_0^{T_*} \|\omega(t)\|_{L^\infty} dt = \infty$ . Subsequently, this result is extended to a larger class of solutions by replacing the  $L^\infty$  norm by the  $BMO$  norm for the vorticity, and  $H^s(\mathbb{R}^n)$  by  $W^{s,p}(\mathbb{R}^n)$  for the velocity in the work [11] of Kozono and Taniuchi. In their continuation work [12], Kozono, Ogawa and Taniuchi further give a sharper criterion, which the  $BMO$  norm is replaced by the Besov space  $\dot{B}_{\infty,\infty}^0$  for the vorticity, since the continuous embedding  $L^\infty \hookrightarrow BMO \hookrightarrow \dot{B}_{\infty,\infty}^0$ . By replacing  $W^{s,p}$  by  $B_{p,r}^s$  for the velocity, Chae [4, 5] establish the criterion by the  $\dot{F}_{\infty,\infty}^0$  norm for the vorticity. Chen, Miao and Zhang [6] also obtained a similar criterion of (1.2) by the  $\dot{F}_{\infty,\infty}^0$  norm for  $\omega$  and  $\nabla \times b$ .

Recently, Based on the weak  $L^p$  spaces  $L^{p,\infty}$ , R. Takada [20] introduced Besov type function spaces  $B_{p,r}^{s,\infty}$ , and established the local well-posedness of solutions for (1.1) in the weak spaces. Following from this line of study, we introduce a class of relatively weaker function spaces, which first initiated by Kozono and Yamazaki [13] to obtain the critical regularity for Navier-Stokes equations. They are modeled on Besov spaces, but the underlying norm is of Morrey type rather

than  $L^p$  or  $L^{p,\infty}$ . One calls them Besov-Morrey spaces. So far, there are few results constructed in the Besov-Morrey spaces for systems (1.1) and (1.2). The main aim of this paper is to answer these problems.

Before stating our main results, we first recall the definitions of Morrey spaces and Besov-Morrey spaces (see [13]) for the convenience of reader.

**Definition 1.1.** For  $1 \leq q \leq p \leq \infty$ , the Morrey space  $M_q^p(\mathbb{R}^n)$  is defined as the set of functions  $f(x) \in L_{loc}^q(\mathbb{R}^n)$  such that

$$\|f\|_{M_q^p(\mathbb{R}^n)} := \sup_{x_0 \in \mathbb{R}^n} \sup_{r>0} r^{n/p-n/q} \left( \int_{B(x_0,r)} |f(y)|^q dy \right)^{1/q} < \infty.$$

Let us remark that it is easy to see that the relation  $M_{q_1}^p(\mathbb{R}^n) \subset M_{q_2}^p(\mathbb{R}^n)$  with  $1 \leq q_2 < q_1 \leq p \leq \infty$ ,  $M_p^p(\mathbb{R}^n) = L^p$  and  $M_q^\infty(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$  for  $1 \leq q \leq \infty$ . In addition, from [13], we know that  $L^{p,\infty}(\mathbb{R}^n) \subset M_q^p(\mathbb{R}^n)$  with  $1 \leq q < p < \infty$ .

Let  $\chi(t) \in C_0^\infty(\mathbb{R})$  such that  $0 \leq \chi(t) \leq 1$ ,  $\chi(t) \equiv 1$  for  $t \leq 1$  and  $\text{supp} \chi \subset [0, 2]$ . Denote  $\mathcal{F}[\varphi_j](\xi) = \chi(2^{-j}|\xi|) - \chi(2^{1-j}|\xi|)$  and  $\mathcal{F}[\varphi_{(0)}](\xi) = \chi(|\xi|)$  for  $\xi \in \mathbb{R}^n$ , where  $\mathcal{F}[g]$  denotes the Fourier transform of  $g$  on  $\mathbb{R}^n$ . Then

$$\mathcal{F}[\varphi_{(0)}](\xi) + \sum_{j \geq 1} \mathcal{F}[\varphi_j](\xi) = 1, \quad \text{for } \xi \in \mathbb{R}^n;$$

$$\sum_{j \in \mathbb{Z}} \mathcal{F}[\varphi_j](\xi) = 1, \quad \text{for } \xi \in \mathbb{R}^n \setminus \{0\}.$$

Given  $f \in \mathcal{S}'$ , where  $\mathcal{S}'$  is the dual space of Schwartz class  $\mathcal{S}$ . To define the homogeneous Besov-Morrey spaces, we set

$$\dot{\Delta}_j f = \mathcal{F}^{-1} \varphi_j(\cdot) \mathcal{F} f, \quad j = 0, \pm 1, \pm 2, \dots$$

**Definition 1.2.** For  $1 \leq q \leq p \leq \infty$ ,  $1 \leq r \leq \infty$ , and  $s \in \mathbb{R}$ , the homogeneous Besov-Morrey space  $\dot{N}_{p,q,r}^s$  is defined by

$$\dot{N}_{p,q,r}^s = \{f \in \mathcal{S}'/\mathcal{P} : \|f\|_{\dot{N}_{p,q,r}^s} < \infty\},$$

where

$$\|f\|_{\dot{N}_{p,q,r}^s} = \begin{cases} \left( \sum_{j \in \mathbb{Z}} (2^{js} \|\dot{\Delta}_j f\|_{M_q^p})^r \right)^{1/r}, & r < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{js} \|\dot{\Delta}_j f\|_{M_q^p}, & r = \infty \end{cases}$$

and  $\mathcal{P}$  denotes the set of polynomials with  $n$  variables.

To define the inhomogeneous Besov-Morrey spaces, we set

$$\Delta_j f = \begin{cases} 0, & j \leq -2, \\ \mathcal{F}^{-1}[\varphi_{(0)}](\xi) \mathcal{F}[f], & j = -1, \\ \mathcal{F}^{-1}[\varphi_j](\xi) \mathcal{F}[f], & j = 0, 1, 2, \dots \end{cases}$$

**Definition 1.3.** For  $1 \leq q \leq p \leq \infty$ ,  $1 \leq r \leq \infty$ , and  $s \in \mathbb{R}$ , the inhomogeneous Besov space  $N_{p,q,r}^s$  is defined by

$$N_{p,q,r}^s = \{f \in \mathcal{S}' : \|f\|_{N_{p,q,r}^s} < \infty\},$$

where

$$\|f\|_{N_{p,q,r}^s} = \begin{cases} \left( \sum_{j=-1}^{\infty} (2^{js} \|\Delta_j f\|_{M_q^p})^r \right)^{1/r}, & r < \infty, \\ \sup_{j \geq -1} 2^{js} \|\Delta_j f\|_{M_q^p}, & r = \infty. \end{cases}$$

It is not difficult to see that the Besov-Morrey space  $\dot{N}_{p,q,r}^s$  and  $N_{p,q,r}^s$  are the Banach spaces. Recall that the standard homogeneous Besov space  $\dot{B}_{p,r}^s$  and inhomogeneous Besov space  $B_{p,r}^s$  (see, e.g., [1]), where the  $L^p$  space is replaced by the Morrey space  $M_q^p$  now.

Main results are stated as follows.

**Theorem 1.1.** (1)(Local-time existence) Let  $1 < q \leq p < \infty$ . Assume that  $s$  and  $r$  satisfy  $s > 1 + n/p$ ,  $1 \leq r \leq \infty$  or  $s = 1 + n/p$ ,  $r = 1$ . Suppose that the initial data  $v_0 \in N_{p,q,r}^s$  satisfying  $\operatorname{div} v_0 = 0$ . Then there exist  $T_1 > 0$  and a unique solution  $v$  of (1.1) such that  $v \in C([0, T_1], N_{p,q,r}^s)$ .  
(2)(Blow-up criterion)

- (i) Let  $s > 1 + n/p$  and  $1 \leq r \leq \infty$ . Then the local-in-time solution  $v \in C([0, T_1], N_{p,q,r}^s)$  blows up at  $T_* > T_1$  in  $N_{p,q,r}^s$ , namely

$$\limsup_{t \rightarrow T_*} \|v(t)\|_{N_{p,q,r}^s} = \infty$$

if only if  $\int_0^{T_*} \|(\nabla \times v)(t)\|_{\dot{B}_{\infty,\infty}^0} dt = \infty$ ;

- (ii) Let  $s = 1 + n/p$  and  $r = 1$ . Then the local-in-time solution  $v \in C([0, T_1], N_{p,q,1}^{1+n/p})$  blows up at  $T_* > T_1$  in  $N_{p,q,1}^{1+n/p}$ , namely

$$\limsup_{t \rightarrow T_*} \|v(t)\|_{N_{p,q,1}^{1+n/p}} = \infty$$

if only if  $\int_0^{T_*} \|(\nabla \times v)(t)\|_{\dot{B}_{\infty,1}^0} dt = \infty$ .

**Remark 1.1.** Since  $L^p(\mathbb{R}^n) \subset L^{p,\infty}(\mathbb{R}^n) \subset M_q^p(\mathbb{R}^n)$ , we have the continuous embeddings  $B_{p,r}^s(\mathbb{R}^n) \hookrightarrow B_{p,r}^{s,\infty}(\mathbb{R}^n) \hookrightarrow N_{p,q,r}^s(\mathbb{R}^n)$ . Therefore, the local existence result contains the previous ones by Chae [5], Takada [20] and Zhou [23]. In addition, due to  $L^\infty(\mathbb{R}^n) \hookrightarrow BMO(\mathbb{R}^n) \hookrightarrow \dot{B}_{\infty,\infty}^0(\mathbb{R}^n) = \dot{F}_{\infty,\infty}^0(\mathbb{R}^n)$ , the blow-up criterion in Theorem 1.1 can be regarded as an improvement of the original Beale-Kato-Majda criterion [2] and a generalization of Chae [5].

For the MHD system (1.2), we have the similar result.

**Theorem 1.2.** (1)(Local-time existence) Let  $1 < q \leq p < \infty$ . Assume that  $s$  and  $r$  satisfy  $s > 1 + n/p$ ,  $1 \leq r \leq \infty$  or  $s = 1 + n/p$ ,  $r = 1$ . Suppose that the initial data  $(v_0, b_0) \in N_{p,q,r}^s$  satisfying  $\operatorname{div} v_0 = 0$  and  $\operatorname{div} b_0 = 0$ . Then there exist  $T_2 > 0$  and a unique solution  $(v, b)$  of (1.2) such that  $(v, b) \in C([0, T_2], N_{p,q,r}^s)$ .  
(2)(Blow-up criterion)

(i) Let  $s > 1 + n/p$  and  $1 \leq r \leq \infty$ . Then the local-in-time solution  $(v, b) \in C([0, T_1], N_{p,q,r}^s)$  blows up at  $T_* > T_2$  in  $N_{p,q,r}^s$ , namely

$$\limsup_{t \rightarrow T_*} \|(v, b)(t)\|_{N_{p,q,r}^s} = \infty$$

if only if  $\int_0^{T_*} (\|\nabla \times v, \nabla \times b\|_{\dot{B}_{\infty,\infty}^0}) dt = \infty$ ;

(ii) Let  $s = 1 + n/p$  and  $r = 1$ . Then the local-in-time solution  $(v, b) \in C([0, T_1], N_{p,q,1}^{1+n/p})$  blows up at  $T_* > T_1$  in  $N_{p,q,1}^{1+n/p}$ , namely

$$\limsup_{t \rightarrow T_*} \|(v, b)(t)\|_{N_{p,q,1}^{1+n/p}} = \infty$$

if only if  $\int_0^{T_*} (\|\nabla \times v, \nabla \times b\|_{\dot{B}_{\infty,1}^0}) dt = \infty$ .

*Remark 1.2.* In the proof of Theorems 1.1-1.2, inspired by [6, 20], we introduce the following particle trajectory mappings

$$\begin{cases} \partial_t X(\alpha, t) = v(X(\alpha, t), t), \\ X(\alpha, 0) = \alpha, \end{cases} \quad (1.3)$$

$$\begin{cases} \partial_t Y(\alpha, t) = (v - b)(Y(\alpha, t), t), \\ Y(\alpha, 0) = \alpha, \end{cases} \quad (1.4)$$

and

$$\begin{cases} \partial_t Z(\alpha, t) = (v + b)(Z(\alpha, t), t), \\ Z(\alpha, 0) = \alpha \end{cases} \quad (1.5)$$

to estimate the frequency-localization solutions to the Euler equations (1.1) and MHD system (1.2) in the  $M_q^p(\mathbb{R}^n)$  space, respectively. It is worth noting that we handle with the coupling effect of the velocity field  $v(x, t)$  and the magnetic field  $b(x, t)$  in (1.2) effectively by (1.4)-(1.5).

Secondly, to deal with frequency-localized nonlinear terms, we develop new commutator estimates in the weaker Besov-Morrey space  $\dot{N}_{p,q,r}^s$  by the Bony's para-product formula. In addition, we should mention the recent preprint [18], where he has announced the partial content (the sup-critical case) of Theorem 1.1, however, the proof of local existence was not available. In fact, by the careful investigation, we think the proof is not obvious. More concretely speaking, according to the work [23] by the second author, we adopt some revised approximate iteration systems instead of that in [4] to construct the local existence of solutions of (1.1) and (1.2).

*Remark 1.3.* Theorems 1.1-1.2 can be regarded as the supplements on the local existence theory of the Euler equations and ideal MHD system. However, the *global* existence of solutions to the 2-dimension case in the framework of Besov-Morrey spaces still remains unsolvable, which is our next consideration.

At the end of Introduction, we also mention other lines of recent study for incompressible Euler equations and MHD system, such as [7, 8] and [24, 25], where they considered the local well-posedness of density-dependent Euler equations and MHD system in several space dimensions.

The rest of this paper unfolds as follows. In Section 2, we briefly review some basic properties of Besov-Morrey spaces. In Section 3, we give some key lemmas. In particular, we develop new

estimates of commutator in Besov-Morrey spaces, which play important roles in the proof of our main theorems. Section 4 is devoted to the total proof of Theorem 1.1. Finally in Section 5, we prove the local well-posedness of MHD system. For brevity, we give the approximate linear system and crucial estimates only.

## 2 Preliminary

Throughout the paper,  $f \lesssim g$  denotes  $f \leq Cg$ , where  $C > 0$  is a generic constant.  $f \approx g$  means  $f \lesssim g$  and  $g \lesssim f$ . In this section, we present some properties in Besov-Morrey spaces defined in Sect. 1 by using the Littlewood-Paley dyadic decomposition. Indeed, Besov-Morrey spaces share many of the properties of Besov spaces, but they represent local oscillations and singularities of functions more precisely. For more details, please refer to [13, 14].

First, we recall the Bernstein's inequality for  $M_q^p$  as for the case of  $L^p$  spaces.

**Lemma 2.1.** *Assume that  $f \in M_q^p(\mathbb{R}^n)$  with  $1 \leq q \leq p \leq \infty$  and  $\text{supp}\mathcal{F}[f] \subset \{2^{j-1} \leq |\xi| < 2^{j+1}\}$ , then there exists a constant  $C_k$  such that the following inequalities holds:*

$$C_k^{-1}2^{jk}\|f\|_{M_q^p} \leq \|D^k f\|_{M_q^p} \leq C_k 2^{jk}\|f\|_{M_q^p}, \quad \text{for all } k \in \mathbb{N},$$

where  $\mathcal{F}[f]$  denotes the usual Fourier transform of  $f$ .

In the Morrey space  $M_q^p$ , since

$$\|\phi * f\|_{M_q^p} \leq C\|\phi\|_{L^1}\|f\|_{M_q^p}$$

holds for  $1 \leq q \leq p \leq \infty$ , we have the immediate consequence of Lemma 2.1.

**Lemma 2.2.** *For  $s \in \mathbb{R}$ ,  $1 \leq q \leq p \leq \infty$ ,  $1 \leq r \leq \infty$ ,  $k \in \mathbb{N}$ , and  $\text{supp}\mathcal{F}[f] \subset \{2^{j-1} \leq |\xi| < 2^{j+1}\}$ , there exists a constant  $C_k$  such that the following inequality holds:*

$$C_k^{-1}\|D^k f\|_{\dot{N}_{p,q,r}^s} \leq \|f\|_{\dot{N}_{p,q,r}^{s+k}} \leq C_k\|D^k f\|_{\dot{N}_{p,q,r}^s}.$$

Next we investigate the relation between the homogeneous and inhomogeneous Besov-Morrey spaces.

**Lemma 2.3.** *For  $s > 0$ ,  $1 \leq q \leq p \leq \infty$ ,  $1 \leq r \leq \infty$ , the following relations hold:*

$$N_{p,q,r}^s(\mathbb{R}^n) = M_q^p(\mathbb{R}^n) \cap \dot{N}_{p,q,r}^s(\mathbb{R}^n),$$

$$\|f\|_{N_{p,q,r}^s} \sim \|f\|_{M_q^p} + \|f\|_{\dot{N}_{p,q,r}^s}.$$

The proof of Lemma 2.3 is standard, see [3] for the similar details. From [13], we have the following Sobolev-type embedding lemma.

**Lemma 2.4.** *For  $s > 0$ ,  $1 \leq q \leq p \leq \infty$ ,  $1 \leq r \leq \infty$ , then*

$$\dot{N}_{p,q,r}^s(\mathbb{R}^n) \hookrightarrow \dot{B}_{\infty,r}^{s-n/p}(\mathbb{R}^n), \quad N_{p,q,r}^s(\mathbb{R}^n) \hookrightarrow B_{\infty,r}^{s-n/p}(\mathbb{R}^n).$$

Since the embedding relation  $\dot{B}_{p,r}^s \hookrightarrow L^\infty, B_{p,r}^s \hookrightarrow L^\infty$  hold for  $s > n/p, 1 \leq p, r \leq \infty$ , or  $s = n/p, 1 \leq p \leq \infty, r = 1$ , we obtain the following conclusion from Lemma 2.4.

**Lemma 2.5.** *Both spaces  $\dot{N}_{p,q,r}^s(\mathbb{R}^n)$  and  $N_{p,q,r}^s(\mathbb{R}^n)$  are Banach algebras for  $s > n/p, 1 \leq q \leq p \leq \infty, r \in [1, \infty]$  or  $s = n/p, 1 \leq q \leq p \leq \infty$  and  $r = 1$ .*

Finally, we present the Bony's para-product formula to end up this section. For simplicity, we state the homogeneous case only.

**Definition 2.1.** *Let  $f, g$  be two temperate distributions. The product  $f \cdot g$  has the Bony's decomposition formally:*

$$f \cdot g = \dot{T}_f g + \dot{T}_g f + \dot{R}(f, g),$$

where  $\dot{T}_f g$  is paraproduct of  $g$  by  $f$ ,

$$\dot{T}_f g = \sum_{j' \leq j-2} \dot{\Delta}_{j'} f \dot{\Delta}_j g = \sum_{j \in \mathbb{Z}} \dot{S}_{j-1} f \dot{\Delta}_j g$$

and the remainder  $\dot{R}(f, g)$  is denoted by

$$\dot{R}(f, g) = \sum_{|j-j'| \leq 1} \dot{\Delta}_j f \dot{\Delta}_{j'} g.$$

The para-product of two temperate distributions is always defined, since the general term of the para-product is spectrally localized in dyadic shells. However, the remainder may not be defined. Roughly speaking, it is defined when  $f$  and  $g$  belong to functional spaces whose sum of regularity index is positive. The reader is referred to [1] for more details on the subject.

### 3 Key lemmas

In this section, we will present some key lemmas, which are used to prove the main results. The first one is related to the particle trajectory mapping.

**Lemma 3.1.** *Assume that  $f \in M_q^p(R^n)$  for  $1 \leq q \leq p \leq \infty$ . If  $X : \alpha \mapsto X(\alpha)$  is a volume-preserving diffeomorphism, then*

$$\|f(\alpha)\|_{M_q^p} = \|f(X(\alpha))\|_{M_q^p}. \quad (3.1)$$

*Proof.* For  $x_0 \in R^n$  and  $r > 0$ ,  $\exists! y_0 \in R^n$  s.t.  $x_0 = X(y_0)$ . Then

$$\begin{aligned} \int_{B(x_0, r)} |f(\alpha)|^q d\alpha &= \int_{X^{-1}(B(x_0, r))} |f(X(\alpha))|^q \det(\nabla_\alpha X(\alpha)) d\alpha \\ &= \int_{X^{-1}(B(x_0, r))} |f(X(\alpha))|^q d\alpha \\ &= \int_{B(y_0, r)} |f(X(\alpha))|^q d\alpha, \end{aligned} \quad (3.2)$$

which implies

$$\sup_{x_0 \in \mathbb{R}^n} \sup_{r > 0} r^{n/p-n/q} \int_{B(x_0, r)} |f(\alpha)|^q d\alpha = \sup_{y_0 \in \mathbb{R}^n} \sup_{r > 0} r^{n/p-n/q} \int_{B(y_0, r)} |f(X(\alpha))|^q d\alpha, \quad (3.3)$$

so (3.1) follows from Definition 1.1 immediately.  $\square$

The next one concerns the logarithmic Besov-Morrey inequality, which is very useful to establish the blow-up criterion in the super-critical case.

**Lemma 3.2.** *[[18]] Let  $s > n/p$  with  $1 \leq q \leq p \leq \infty$ ,  $1 \leq r \leq \infty$ . Assume  $f \in N_{p,q,r}^s$ , then there exists a constant  $C$  such that the following inequality holds:*

$$\|f\|_{L^\infty} \leq C \left( 1 + \|f\|_{\dot{B}_{\infty,\infty}^0} (\log^+ \|f\|_{N_{p,q,r}^s} + 1) \right). \quad (3.4)$$

In order to estimate the bilinear terms, we need the following Moser-type inequalities in Besov-Morrey spaces.

**Lemma 3.3.** *[[18]] Let  $s > n/p$  with  $1 \leq q \leq p < \infty$ ,  $1 \leq r \leq \infty$  or  $p = r = \infty$ . Then exists a constant  $C$  such that the following inequalities hold:*

$$\|fg\|_{\dot{N}_{p,q,r}^s} \leq C \left( \|f\|_{M_{q_1}^{p_1}} \|g\|_{\dot{N}_{p_2,q_2,r}^s} + \|g\|_{M_{q_3}^{p_3}} \|f\|_{\dot{N}_{p_4,q_4,r}^s} \right), \quad (3.5)$$

$$\|fg\|_{N_{p,q,r}^s} \leq C \left( \|f\|_{M_{q_1}^{p_1}} \|g\|_{N_{p_2,q_2,r}^s} + \|g\|_{M_{q_3}^{p_3}} \|f\|_{N_{p_4,q_4,r}^s} \right), \quad (3.6)$$

and

$$\|fg\|_{\dot{N}_{p,q,r}^s} \leq C \left( \|f\|_{\dot{N}_{p_1,q_1,r_1}^{-\alpha}} \|g\|_{\dot{N}_{p_2,q_2,r_2}^{s+\alpha}} + \|g\|_{\dot{N}_{p_3,q_3,r_3}^{-\alpha}} \|f\|_{\dot{N}_{p_4,q_4,r_4}^{s+\alpha}} \right) \quad (3.7)$$

for  $\alpha > 0$ , where  $1 \leq q_1 \leq p_1 \leq \infty$  and  $1 \leq q_3 \leq p_3 \leq \infty$ , such that

$$1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4, \quad 1/r = 1/r_1 + 1/r_2 = 1/r_3 + 1/r_4,$$

$$1/q \leq 1/q_1 + 1/q_2, \quad \text{and} \quad 1/q \leq 1/q_3 + 1/q_4.$$

The last one concerns the commutator estimates, which plays the important role in the treatment of frequency-localized nonlinear terms.

**Lemma 3.4.** *For  $s > 0$ ,  $1 \leq q \leq p < \infty$  and  $1 \leq r \leq \infty$ , there is a constant  $C$  such that*

$$\|2^{sj} [[v \cdot \nabla, \dot{\Delta}_j] \theta]_{M_q^p} \|_{\ell^r} \leq C \left( \|\nabla v\|_{L^\infty} \|\theta\|_{\dot{N}_{p,q,r}^s} + \|\nabla \theta\|_{M_{q_1}^{p_1}} \|v\|_{N_{p_2,q_2,r}^s} \right) \quad (3.8)$$

holds for all  $\theta \in \dot{N}_{p,q,r}^s$  with  $\nabla \theta \in M_{q_1}^{p_1}$  and all  $v \in N_{p_2,q_2,r}^s$  with  $\nabla v \in L^\infty$  such that  $\operatorname{div} v = 0$ , where  $1 \leq q_1 \leq p_1 \leq \infty$  such that  $1/p = 1/p_1 + 1/p_2$  and  $1/q \leq 1/q_1 + 1/q_2$ .



*Proof.* By Bony's para-product, we decompose  $[v \cdot \nabla, \dot{\Delta}_j]\theta = K_1 + K_2 + K_3 + K_4 + K_5$  with

$$K_1 = [T_{v^i} \partial_i, \dot{\Delta}_j]\theta, \quad K_2 = -\dot{\Delta}_j T_{\partial_i \theta} v^i, \quad K_3 = T_{\partial_i \dot{\Delta}_j \theta} v^i,$$

$$K_4 = -\dot{\Delta}_j R(v^i, \partial_i \theta), \quad K_5 = R(v^i, \partial_i \dot{\Delta}_j \theta),$$

where the Einstein notation was used for simplicity.

From the definition of  $\dot{\Delta}_j$ , we have the almost orthogonal properties:

$$\dot{\Delta}_i \dot{\Delta}_j f \equiv 0 \quad \text{if } |i - j| \geq 2,$$

$$\dot{\Delta}_j (\dot{S}_{j-1} f \dot{\Delta}_i g) \equiv 0 \quad \text{if } |i - j| \geq 5.$$

For  $K_1$ , it follows from the fact  $\operatorname{div} \dot{S}_{j-1} v = 0$  for all  $j \in \mathbb{Z}$  and orthogonal properties that

$$\begin{aligned} K_1 &= T_{v^i} \partial_i \dot{\Delta}_j \theta - \dot{\Delta}_j T_{v^i} \partial_i \theta \\ &= \sum_{j' \in \mathbb{Z}} \left\{ S_{j'-1} v^i \dot{\Delta}_j (\partial_i \dot{\Delta}_{j'} \theta) - \dot{\Delta}_j (S_{j'-1} v^i \dot{\Delta}_{j'} \partial_i \theta) \right\} \\ &= \sum_{|j-j'| \leq 4} 2^{jn} \int_{\mathbb{R}^n} \varphi_0(2^j(x-y)) \left\{ S_{j'-1} v^i(x) - S_{j'-1} v^i(y) \right\} \partial_i \dot{\Delta}_{j'} \theta(y) dy \\ &= \sum_{|j-j'| \leq 4} 2^{j(n+1)} \int_{\mathbb{R}^n} \partial_i \varphi_0(2^j(x-y)) \left\{ S_{j'-1} v^i(x) - S_{j'-1} v^i(y) \right\} \dot{\Delta}_{j'} \theta(y) dy \\ &= \sum_{|j-j'| \leq 4} 2^{j(n+1)} \int_{\mathbb{R}^n} \partial_i \varphi_0(2^j(x-y)) \int_0^1 ((x-y) \cdot \nabla) S_{j'-1} v^i(x + \tau(y-x)) d\tau \dot{\Delta}_{j'} \theta(y) dy \\ &= \sum_{|j-j'| \leq 4} \int_{\mathbb{R}^n} \partial_i \varphi_0(z) \int_0^1 (z \cdot \nabla) S_{j'-1} v^i(x - \tau 2^{-j} z) d\tau \dot{\Delta}_{j'} \theta(x - 2^{-j} z) dz, \end{aligned}$$

where we have performed the integration by parts. Then, by Young's inequality in Morrey spaces, we obtain

$$\begin{aligned} \|K_1\|_{M_q^p} &\leq C \|\nabla v\|_{L^\infty} \sum_{|j-j'| \leq 4} \left\| \int_{\mathbb{R}^n} |z \cdot \nabla \varphi(z)| |\dot{\Delta}_{j'} \theta(\cdot - 2^{-j} z)| dz \right\|_{M_q^p} \\ &\leq C \|\nabla v\|_{L^\infty} \sum_{|j-j'| \leq 4} \|\dot{\Delta}_{j'} \theta\|_{M_q^p}. \end{aligned} \tag{3.9}$$

For  $K_2$ , it is clear that

$$K_2 = - \sum_{|j-j'| \leq 4} \dot{\Delta}_j \left\{ (S_{j'-1} \partial_i \theta) (\dot{\Delta}_{j'} v^i) \right\}$$

which implies that

$$\begin{aligned} \|K_2\|_{M_q^p} &\leq C \sum_{|j-j'| \leq 4} \|S_{j'-1} \partial_i \theta\|_{M_{q_1}^{p_1}} \|\dot{\Delta}_{j'} v^i\|_{M_{q_2}^{p_2}} \\ &\leq C \|\nabla \theta\|_{M_{q_1}^{p_1}} \sum_{|j-j'| \leq 4} \|\dot{\Delta}_{j'} v\|_{M_{q_2}^{p_2}}, \end{aligned} \tag{3.10}$$

where we used the Hölder inequality in Morrey spaces with  $1/p = 1/p_1 + 1/p_2$ ,  $1/q \leq 1/q_1 + 1/q_2$  and  $1 \leq q_1 \leq p_1 \leq \infty$ .

For  $K_3$ , note that  $S_{j'-1} \partial_i \dot{\Delta}_j \theta = 0$ , if  $j' \leq j$ , we may rewrite

$$K_3 = \sum_{j' \geq j+1} (S_{j'-1} \partial_i \dot{\Delta}_j \theta) (\dot{\Delta}_{j'} v^i),$$

Then the Hölder inequality gives

$$\begin{aligned} \|K_3\|_{M_q^p} &\leq \sum_{j' \geq j+1} \|S_{j'-1} \partial_i \dot{\Delta}_j \theta\|_{M_{q_1}^{p_1}} \|\dot{\Delta}_{j'} v^i\|_{M_{q_2}^{p_2}} \\ &\leq C \|\nabla \theta\|_{M_{q_1}^{p_1}} \sum_{j' \geq j+1} \|\dot{\Delta}_{j'} v\|_{M_{q_2}^{p_2}}. \end{aligned} \quad (3.11)$$

For  $K_4$ , by the definition of  $\dot{R}$ , we may rewrite

$$\begin{aligned} K_4 &= -\dot{\Delta}_j \left\{ \sum_{j' \in \mathbb{Z}} \sum_{|j-j'| \leq 1} (\dot{\Delta}_{j'} v^i) (\dot{\Delta}_{j''} \partial_i \theta) \right\} \\ &= - \sum_{\max(j', j'') \geq j-2} \sum_{|j-j'| \leq 1} \dot{\Delta}_j \{ (\dot{\Delta}_{j'} v^i) (\dot{\Delta}_{j''} \partial_i \theta) \}, \end{aligned}$$

which yields

$$\begin{aligned} \|K_4\|_{M_q^p} &\leq \sum_{\max(j', j'') \geq j-2} \sum_{|j-j'| \leq 1} \|\dot{\Delta}_{j''} \partial_i \theta\|_{M_{q_1}^{p_1}} \|\dot{\Delta}_{j'} v^i\|_{M_{q_2}^{p_2}} \\ &\leq C \|\nabla \theta\|_{M_{q_1}^{p_1}} \sum_{j' \geq j-2} \|\dot{\Delta}_{j'} v\|_{M_{q_2}^{p_2}}. \end{aligned} \quad (3.12)$$

For the last term  $K_5$ , it holds that

$$\begin{aligned} K_5 &= \sum_{j' \in \mathbb{Z}} \sum_{|j-j'| \leq 1} (\dot{\Delta}_{j'} v^i) (\dot{\Delta}_{j''} \partial_i \dot{\Delta}_j \theta) \\ &= \sum_{|j-j'| \leq 2} \sum_{|j-j'| \leq 1} \{ (\dot{\Delta}_{j'} v^i) (\dot{\Delta}_{j''} \partial_i \dot{\Delta}_j \theta) \}. \end{aligned}$$

Then, we get

$$\begin{aligned} \|K_5\|_{M_q^p} &\leq \sum_{|j-j'| \leq 2} \sum_{|j-j'| \leq 1} \|\dot{\Delta}_{j''} \partial_i \dot{\Delta}_j \theta\|_{M_{q_1}^{p_1}} \|\dot{\Delta}_{j'} v^i\|_{M_{q_2}^{p_2}} \\ &\leq C \|\nabla \theta\|_{M_{q_1}^{p_1}} \sum_{j' \geq j-2} \|\dot{\Delta}_{j'} v\|_{M_{q_2}^{p_2}}. \end{aligned} \quad (3.13)$$

Together with these estimates (3.9)-(3.13), we conclude that

$$\begin{aligned} \|[v \cdot \nabla, \dot{\Delta}_j] \theta\|_{M_q^p} &\leq C \|\nabla v\|_{L^\infty} \sum_{|j-j'| \leq 4} \|\dot{\Delta}_{j'} \theta\|_{M_q^p} + C \|\nabla \theta\|_{M_{q_1}^{p_1}} \sum_{|j-j'| \leq 4} \|\dot{\Delta}_{j'} v\|_{M_{q_2}^{p_2}} \\ &\quad + C \|\nabla \theta\|_{M_{q_1}^{p_1}} \sum_{j' \geq j-2} \|\dot{\Delta}_{j'} v\|_{M_{q_2}^{p_2}}. \end{aligned} \quad (3.14)$$

Finally, we apply the Young inequality for sequences to get (3.8) immediately, where  $s > 0$  is required. Thus we complete the proof of Lemma 3.4.  $\square$

**Lemma 3.5.** *For  $s > -1$ ,  $1 \leq q \leq p < \infty$  and  $1 \leq r \leq \infty$ , there is a constant  $C$  such that*

$$\|2^{js}\| [v \cdot \nabla, \dot{\Delta}_j] \theta \|_{M_q^p} \ell^r \leq C \left( \|\nabla v\|_{L^\infty} \|\theta\|_{\dot{N}_{p,q,r}^s} + \|\theta\|_{M_{q_1}^{p_1}} \|v\|_{\dot{N}_{p_2,q_2,r}^{s+1}} \right) \quad (3.15)$$

holds for all  $\theta \in \dot{N}_{p,q,r}^s \cap M_{q_1}^{p_1}$  and all  $v \in \dot{N}_{p_2,q_2,r}^{s+1}$  with  $\nabla v \in L^\infty$  such that  $\operatorname{div} v = 0$ , where  $1 \leq q_1 \leq p_1 \leq \infty$  such that  $1/p = 1/p_1 + 1/p_2$ ,  $1/q \leq 1/q_1 + 1/q_2$ .

*Proof.* As in the proof of Lemma 3.4, we decompose

$$[v \cdot \nabla, \dot{\Delta}_j] \theta := K_1 + K_2 + K_3 + K_4 + K_5,$$

The estimate of  $K_1$  is still valid, however, different estimates are needed for  $K_2, K_3, K_4$  and  $K_5$ .

For  $K_2$ , since  $\operatorname{div} \dot{\Delta}_{j'} v = 0$  for all  $j' \in \mathbb{Z}$ , by integration by parts, we have

$$\begin{aligned} K_2 &= - \sum_{|j-j'| \leq 4} \dot{\Delta}_j \left\{ (S_{j'-1} \partial_i \theta) (\dot{\Delta}_{j'} v^i) \right\} \\ &= - \sum_{|j-j'| \leq 4} 2^{jn} \int_{R^n} \varphi_0(2^j(x-y)) (S_{j'-1} \partial_i \theta)(y) \dot{\Delta}_{j'} v^i(y) dy \\ &= - \sum_{|j-j'| \leq 4} 2^j 2^{jn} \int_{R^n} \partial_i \varphi_0(2^j(x-y)) (S_{j'-1} \theta)(y) \dot{\Delta}_{j'} v^i(y) dy \\ &= - \sum_{|j-j'| \leq 4} 2^j \int_{R^n} \partial_i \varphi_0(z) S_{j'-1} \theta(x - 2^{-j} z) \dot{\Delta}_{j'} v^i(x - 2^{-j} z) dz. \end{aligned}$$

Then by Young's and Hölder inequalities in Morrey spaces, we obtain

$$\begin{aligned} \|K_2\|_{M_q^p} &\leq \sum_{|j-j'| \leq 4} 2^j \left\| \int_{R^n} \partial_i \varphi_0(z) S_{j'-1} \theta(\cdot - 2^{-j} z) \dot{\Delta}_{j'} v^i(\cdot - 2^{-j} z) dz \right\|_{M_q^p} \\ &\leq C \sum_{|j-j'| \leq 4} 2^j \| (S_{j'-1} \theta) (\dot{\Delta}_{j'} v^i) \|_{M_q^p} \\ &\leq C \|\theta\|_{M_{q_1}^{p_1}} \sum_{|j-j'| \leq 4} 2^j \|\dot{\Delta}_{j'} v^i\|_{M_{q_2}^{p_2}}. \end{aligned} \quad (3.16)$$

Using the Bernstein's inequality, we proceed  $K_3$  as follows:

$$\begin{aligned} \|K_3\|_{M_q^p} &\leq \sum_{j' \geq j+1} \|S_{j'-1} \partial_i \dot{\Delta}_j \theta\|_{M_{q_1}^{p_1}} \|\dot{\Delta}_{j'} v^i\|_{M_{q_2}^{p_2}} \\ &\leq C \sum_{j' \geq j+1} 2^j \|\dot{\Delta}_j \theta\|_{M_{q_1}^{p_1}} \|\dot{\Delta}_{j'} v^i\|_{M_{q_2}^{p_2}} \\ &\leq C \|\theta\|_{M_{q_1}^{p_1}} \sum_{j' \geq j+1} 2^j \|\dot{\Delta}_{j'} v^i\|_{M_{q_2}^{p_2}}. \end{aligned} \quad (3.17)$$

For  $K_4$ , we have by integration by parts

$$\begin{aligned}
K_4 &= - \sum_{\max(j', j'') \geq j-2} \sum_{|j'-j''| \leq 1} 2^{jn} \int_{R^n} \varphi_0(2^j(x-y)) (\dot{\Delta}_{j'} v^i)(y) (\dot{\Delta}_{j''} \partial_i \theta)(y) dy \\
&= - \sum_{\max(j', j'') \geq j-2} \sum_{|j'-j''| \leq 1} 2^j 2^{jn} \int_{R^n} \partial_i \varphi_0(2^j(x-y)) (\dot{\Delta}_{j'} v^i)(y) (\dot{\Delta}_{j''} \theta)(y) dy \\
&= - \sum_{\max(j', j'') \geq j-2} \sum_{|j'-j''| \leq 1} 2^j \int_{R^n} \partial_i \varphi_0(z) (\dot{\Delta}_{j'} v^i)(x-2^{-j}z) (\dot{\Delta}_{j''} \theta)(x-2^{-j}z) dz,
\end{aligned}$$

which leads to

$$\begin{aligned}
\|K_4\|_{M_q^p} &\leq \sum_{\max(j', j'') \geq j-2} \sum_{|j'-j''| \leq 1} 2^j \|(\dot{\Delta}_{j'} v^i)(\dot{\Delta}_{j''} \theta)\|_{M_q^p} \\
&\leq C \|\theta\|_{M_{q_1}^{p_1}} \sum_{j' \geq j-2} 2^j \|\dot{\Delta}_{j'} v\|_{M_{q_2}^{p_2}}.
\end{aligned} \tag{3.18}$$

For the estimate of  $K_5$ , we recall

$$K_5 = \sum_{|j-j'| \leq 2} \sum_{|j-j''| \leq 1} \{(\dot{\Delta}_{j'} v^i)(\dot{\Delta}_{j''} \partial_i \dot{\Delta}_j \theta)\}.$$

furthermore, we get

$$\begin{aligned}
\|K_5\|_{M_q^p} &\leq \sum_{|j-j'| \leq 2} \sum_{|j-j''| \leq 1} \|\dot{\Delta}_{j''} \partial_i \dot{\Delta}_j \theta\|_{M_{q_1}^{p_1}} \|\dot{\Delta}_{j'} v^i\|_{M_{q_2}^{p_2}} \\
&\leq C \sum_{|j-j'| \leq 2} 2^j \|\dot{\Delta}_j \theta\|_{M_{q_1}^{p_1}} \|\dot{\Delta}_{j'} v\|_{M_{q_2}^{p_2}} \\
&\leq C \|\theta\|_{M_{q_1}^{p_1}} \sum_{|j-j'| \leq 2} 2^j \|\dot{\Delta}_{j'} v\|_{M_{q_2}^{p_2}}.
\end{aligned} \tag{3.19}$$

From these inequalities (3.9), (3.16)-(3.19), we are led to

$$\begin{aligned}
\|[v \cdot \nabla, \dot{\Delta}_j] \theta\|_{M_q^p} &\leq C \|\nabla v\|_{L^\infty} \sum_{|j-j'| \leq 4} \|\dot{\Delta}_{j'} \theta\|_{M_q^p} + C \|\theta\|_{M_{q_1}^{p_1}} \sum_{|j-j'| \leq 4} 2^j \|\dot{\Delta}_{j'} v\|_{M_{q_2}^{p_2}} \\
&\quad + C \|\theta\|_{M_{q_1}^{p_1}} \sum_{j' \geq j-2} 2^j \|\dot{\Delta}_{j'} v\|_{M_{q_2}^{p_2}},
\end{aligned} \tag{3.20}$$

which implies that

$$\|2^{js} \|[v \cdot \nabla, \dot{\Delta}_j] \theta\|_{M_q^p}\|_{\ell^r} \leq C \left( \|\nabla v\|_{L^\infty} \|\theta\|_{\dot{N}_{p,q,r}^s} + \|\theta\|_{M_{q_1}^{p_1}} \|v\|_{\dot{N}_{p_2,q_2,r}^{s+1}} \right) \tag{3.21}$$

where  $s+1 > 0$  is required. This just the inequality (3.15). Hence the proof of Lemma 3.5 is finished.  $\square$

## 4 Proof of Theorem 1.1

In this section, we begin to prove Theorem 1.1 with the aid of key Lemmas 3.1-3.5. The proof is divided into several steps, since it is a bit longer.

Step 1: The linear equation of (1.1)

Consider the linear transport system as in [23]:

$$\begin{cases} \partial_t v + (w \cdot \nabla)v + \nabla P = 0, \\ \operatorname{div} v = 0, \\ v(x, 0) = v_0(x). \end{cases} \quad (4.1)$$

Then we have following local existence result for (4.1), which will be proved in the last step.

**Proposition 4.1.** *Assume that  $\operatorname{div} w = 0, w \in L^\infty(0, T, N_{p,q,r}^s)$  for some  $T > 0, s > 1 + n/p, 1 < q \leq p < \infty, r \in [1, \infty]$  or  $s = 1 + n/p, 1 < q \leq p < \infty$  and  $r = 1$ . Then for any  $v_0 \in N_{p,q,r}^s$  satisfying  $\operatorname{div} v_0 = 0$ , there exists a unique solution  $v \in C([0, T]; N_{p,q,r}^s)$  to the linear system (4.1). And consequently,  $\nabla P$  can be determined uniquely.*

Step 2: Approximate solutions and uniform estimates

The proof of main theorem depends on the standard iteration argument. To obtain the approximate solutions, we first set  $v^0 = 0$  and then define  $\{v^{m+1}\}$  as the solutions of the following linear system

$$\begin{cases} \partial_t v^{m+1} + (v^m \cdot \nabla)v^{m+1} + \nabla P^{m+1} = 0, \\ \operatorname{div} v^{m+1} = 0, \quad \operatorname{div} v^m = 0, \\ v^{m+1}(x, 0) = S_{m+1}v_0(x), \end{cases} \quad (4.2)$$

for  $m = 0, 1, 2, \dots$ . In [4], Chae took a similar (not same) iterative system to construct the local solution. But unfortunately, the linear system (3.32)-(3.33) on p.671 of [4] is unsolvable, since the system itself lacks consistence.

If we have the uniform estimate for the sequence  $\{v^m\}$  by induction, which satisfies the assumptions in Proposition 4.1, then the system (4.2) can be solved with solution  $\{v^{m+1}\}$ .

For that purpose, we turn to derive the uniform estimates of solutions. Applying the homogeneous operator  $\dot{\Delta}_j (j \in \mathbb{Z})$  to the first equation of (4.2), we have

$$\partial_t \dot{\Delta}_j v^{m+1} + (v^m \cdot \nabla) \dot{\Delta}_j v^{m+1} = [v^m \cdot \nabla, \dot{\Delta}_j] v^{m+1} - \dot{\Delta}_j \nabla P^{m+1}. \quad (4.3)$$

Define by  $\{X^m(\alpha, t)\}$  the family of particle trajectory mapping as follows

$$\begin{cases} \partial_t X^m(\alpha, t) = v^m(X^m(\alpha, t), t), \\ X^m(\alpha, 0) = \alpha. \end{cases} \quad (4.4)$$

Note that  $\operatorname{div} v^m = 0$  implies that each  $\alpha \mapsto X^m(\alpha, t)$  is a volume-preserving mapping for all  $t > 0$ . It follows from the particle trajectory mapping that

$$\partial_t \dot{\Delta}_j v^{m+1} + (v^m \cdot \nabla) \dot{\Delta}_j v^{m+1} \Big|_{(x,t)=(X^m(\alpha,t),t)} = \frac{\partial}{\partial t} \dot{\Delta}_j v^{m+1}(X^m(\alpha, t), t) \quad (4.5)$$

which gives

$$\begin{aligned}
|\dot{\Delta}_j v^{m+1}(X^m(\alpha, t), t)| &\leq |\dot{\Delta}_j v^{m+1}(\alpha, 0)| + \int_0^t |\dot{\Delta}_j \nabla P^{m+1}(X^m(\alpha, \tau), \tau)| d\tau \\
&+ \int_0^t |[v^m \cdot \nabla, \dot{\Delta}_j] v^{m+1}(X^m(\alpha, \tau), \tau)| d\tau.
\end{aligned} \tag{4.6}$$

Taking the  $M_q^p$  norm ( $1 \leq q < p < \infty$ ) on both sides of (4.6), with the help of Lemma 3.1, we get

$$\begin{aligned}
\|\dot{\Delta}_j v^{m+1}(t)\|_{M_q^p} &\leq \|\dot{\Delta}_j v_0^{m+1}\|_{M_q^p} + C \int_0^t \|\dot{\Delta}_j \nabla P^{m+1}(\tau)\|_{M_q^p} d\tau \\
&+ C \int_0^t \|[v^m \cdot \nabla, \dot{\Delta}_j] v^{m+1}(\tau)\|_{M_q^p} d\tau.
\end{aligned} \tag{4.7}$$

Then, we multiply both sides by  $2^{js}$  and take the  $\ell^r$  norm, and use Minkowski's inequality to obtain

$$\begin{aligned}
\|v^{m+1}(t)\|_{\dot{N}_{p,q,r}^s} &\leq \|v_0^{m+1}\|_{\dot{N}_{p,q,r}^s} + C \int_0^t \|\nabla P^{m+1}(\tau)\|_{\dot{N}_{p,q,r}^s} d\tau \\
&+ C \int_0^t \|2^{js} \|[v^m \cdot \nabla, \dot{\Delta}_j] v^{m+1}(\tau)\|_{M_q^p}\|_{\ell^r} d\tau.
\end{aligned} \tag{4.8}$$

Thanks to the commutator estimate in Lemma 3.4, by taking  $p_1 = \infty$  and  $q_1 = q_2 = q$ , we have

$$\begin{aligned}
&\|2^{js} \|[v^m \cdot \nabla, \dot{\Delta}_j] v^{m+1}(\tau)\|_{M_q^p}\|_{\ell^r} \\
&\leq C \left( \|\nabla v^m\|_{L^\infty} \|v^{m+1}\|_{\dot{N}_{p,q,r}^s} + \|\nabla v^{m+1}\|_{L^\infty} \|v^m\|_{\dot{N}_{p,q,r}^s} \right) \\
&\leq C \|v^m\|_{\dot{N}_{p,q,r}^s} \|v^{m+1}\|_{\dot{N}_{p,q,r}^s},
\end{aligned} \tag{4.9}$$

where we have used the Sobolev embedding relations  $\dot{N}_{p,q,r}^{s-1} \hookrightarrow L^\infty$  for  $s > n/p + 1, 1 \leq q \leq p < \infty, 1 \leq r \leq \infty$  or  $s = n/p + 1, 1 \leq q \leq p < \infty$  and  $r = 1$ .

Next we turn our attention to the estimates for the pressure term. Taking the divergence on both sides of (4.2), we have

$$-\Delta P^{m+1} = \operatorname{div}(v^m \cdot \nabla) v^{m+1} \tag{4.10}$$

which implies

$$\partial_i \partial_j P^{m+1} = R_i R_j \operatorname{div}(v^m \cdot \nabla) v^{m+1}, \tag{4.11}$$

where  $R_i (i = 1, 2, \dots, n)$  are the  $n$ -dimensional Riesz transform. Since  $\operatorname{div} v^m = 0$ , we obtain

$$\operatorname{div}(v^m \cdot \nabla) v^{m+1} = \sum_{k,l=1}^n \partial_k v_l^m \partial_l v_k^{m+1} = \sum_{k,l=1}^n \partial_l (\partial_k v_l^m v_k^{m+1}). \tag{4.12}$$

Thus, by Bernstein's lemma, we arrive at

$$\begin{aligned}
\|\nabla P^{m+1}\|_{\dot{N}_{p,q,r}^s} &\leq C \sum_{i,j=1}^n \|\partial_i \partial_j P^{m+1}\|_{\dot{N}_{p,q,r}^{s-1}} \\
&\leq C \sum_{i,j,k,l=1}^n \|R_i R_j \partial_k v_l^m \partial_l v_k^{m+1}\|_{\dot{N}_{p,q,r}^{s-1}} \\
&\leq C \sum_{k,l=1}^n \|\partial_k v_l^m \partial_l v_k^{m+1}\|_{\dot{N}_{p,q,r}^{s-1}} \\
&\leq C \|\nabla v^m\|_{L^\infty} \|\nabla v^{m+1}\|_{\dot{N}_{p,q,r}^{s-1}} + \|\nabla v^{m+1}\|_{L^\infty} \|\nabla v^m\|_{\dot{N}_{p,q,r}^{s-1}} \\
&\leq C \|v^m\|_{\dot{N}_{p,q,r}^s} \|v^{m+1}\|_{\dot{N}_{p,q,r}^s},
\end{aligned} \tag{4.13}$$

where we have taken  $p_1 = p_3 = \infty$ ,  $p_2 = p_4 = p$  and  $q_1 = q_3 = \infty$ ,  $q_2 = q_4 = q$  in Lemma 3.3.

It follows from (4.8), (4.9) and (4.13) that

$$\begin{aligned}
\|v^{m+1}(t)\|_{\dot{N}_{p,q,r}^s} &\leq \|v_0^{m+1}\|_{\dot{N}_{p,q,r}^s} + C \int_0^t \|v^m(\tau)\|_{N_{p,q,r}^s} \|v^{m+1}(\tau)\|_{N_{p,q,r}^s} d\tau \\
&\leq C \|v_0\|_{\dot{N}_{p,q,r}^s} + C \int_0^t \|v^m(\tau)\|_{N_{p,q,r}^s} \|v^{m+1}(\tau)\|_{N_{p,q,r}^s} d\tau.
\end{aligned} \tag{4.14}$$

Moreover, in order to show the estimate in the inhomogeneous Besov-Morrey spaces  $N_{p,q,r}^s$ , we need to bound  $\|v^{m+1}(t)\|_{M_q^p}$ . Similarly, we have

$$|v^{m+1}(X^m(\alpha, t), t)| \leq |v^{m+1}(\alpha, 0)| + \int_0^t |\nabla P^{m+1}(X^m(\alpha, \tau), \tau)| d\tau. \tag{4.15}$$

Furthermore, from Lemma 3.1, we get

$$\|v^{m+1}(t)\|_{M_q^p} \leq \|v^{m+1}(0)\|_{M_q^p} + C \int_0^t \|\nabla P^{m+1}(\tau)\|_{M_q^p} d\tau, \tag{4.16}$$

where the pressure can be estimate as follows

$$\begin{aligned}
\|\nabla P^{m+1}\|_{M_q^p} &\leq C \sum_{k=1}^n \left\| \nabla(-\Delta)^{-1} \partial_k \left\{ (v^m \cdot \nabla) v_k^{m+1} \right\} \right\|_{M_q^p} \\
&\leq C \|(v^m \cdot \nabla) v_k^{m+1}\|_{M_q^p} \\
&\leq C \|v^m\|_{M_q^p} \|\nabla v^{m+1}\|_{L^\infty} \\
&\leq C \|v^m\|_{N_{p,q,r}^s} \|v^{m+1}\|_{N_{p,q,r}^s}
\end{aligned} \tag{4.17}$$

Substituting (4.17) into (4.16), we have

$$\|v^{m+1}(t)\|_{M_q^p} \leq C \|v(0)\|_{M_q^p} + C \int_0^t \|v^m(\tau)\|_{N_{p,q,r}^s} \|v^{m+1}(\tau)\|_{N_{p,q,r}^s} d\tau. \tag{4.18}$$

Therefore, adding (4.14) to (4.18) together, by Lemma 2.3, we are led to the inhomogeneous space estimate

$$\|v^{m+1}(t)\|_{N_{p,q,r}^s} \leq C\|v_0\|_{N_{p,q,r}^s} + C \int_0^t \|v^m(\tau)\|_{N_{p,q,r}^s} \|v^{m+1}(\tau)\|_{N_{p,q,r}^s} d\tau. \quad (4.19)$$

It follows from Gronwall's inequality that

$$\|v^{m+1}(t)\|_{N_{p,q,r}^s} \leq C\|v_0\|_{N_{p,q,r}^s} \exp\left(C \int_0^t \|v^m(\tau)\|_{N_{p,q,r}^s} d\tau\right), \quad (4.20)$$

where the generic constant  $C > 0$  maybe depend on  $n$  and  $p, q$ , but it is independent of  $m$ . Therefore we can obtain the uniform estimates by induction.

In fact, we take  $C_1 > 0$  such that

$$\|v_0\|_{N_{p,q,r}^s} \leq \frac{C_1}{2C},$$

then the following inequality holds

$$\|v^m\|_{L_{T_1}^\infty(N_{p,q,r}^s)} \leq C_1, \quad (4.21)$$

for all  $m \geq 0$ , provided that  $T_1 > 0$  (independent of  $m$ ) is sufficiently small.

(4.21) can be shown easily by the standard induction. First, it is true to for  $m = 0$ . Suppose (4.21) holds for  $m > 0$ , it follows from (4.20) that

$$\|v^{m+1}(t)\|_{N_{p,q,r}^s} \leq \frac{C_1}{2} \exp\left(C \int_0^T \|v^m(\tau)\|_{N_{p,q,r}^s} d\tau\right) \leq \frac{C_1}{2} \exp(CC_1T), \quad \text{for } t \in [0, T]. \quad (4.22)$$

Hence, (4.21) holds, if we choose  $T_1 > 0$  so small that  $\exp(CC_1T_1) \leq 2$ . Moreover,  $T_1$  is independent of  $m$ .

### Step 3: Convergence and existence

To prove the convergence, it is sufficient to estimate the difference of the iteration. Set

$$u^{m+1} = v^{m+1} - v^m, \quad \nabla \Pi^{m+1} = \nabla P^{m+1} - \nabla P^m.$$

Then we take the difference between the equation (4.2) for the  $(m+1)$ -th step and the  $m$ -th step to get

$$\begin{cases} \partial_t u^{m+1} + (v^m \cdot \nabla) u^{m+1} + (u^m \cdot \nabla) v^m + \nabla \Pi^{m+1} = 0, \\ \operatorname{div} u^{m+1} = 0, \quad \operatorname{div} v^m = 0, \\ u^{m+1}(x, 0) = S_{m+1} v_0(x) - S_m v_0(x) = \Delta_m v_0(x). \end{cases} \quad (4.23)$$

Taking  $\dot{\Delta}_j (j \in \mathbb{Z})$  on the first equation of (4.23), we obtain

$$\partial_t \dot{\Delta}_j u^{m+1} + (v^m \cdot \nabla) \dot{\Delta}_j u^{m+1} = [v^m \cdot \nabla, \dot{\Delta}_j] u^{m+1} - \dot{\Delta}_j ((u^m \cdot \nabla) v^m) - \dot{\Delta}_j \nabla \Pi^{m+1}. \quad (4.24)$$



By the definition of  $X^m$ , similar to (4.6), we arrive at

$$\begin{aligned}
& |\dot{\Delta}_j u^{m+1}(X^m(\alpha, t), t)| \\
& \leq |\dot{\Delta}_j u^{m+1}(\alpha, 0)| + \int_0^t |[v^m \cdot \nabla, \dot{\Delta}_j] u^{m+1}(X^m(\alpha, \tau), \tau)| d\tau \\
& \quad + \int_0^t |\dot{\Delta}_j((u^m \cdot \nabla)v^m)(X^m(\alpha, \tau), \tau)| d\tau + \int_0^t |\dot{\Delta}_j \nabla \Pi^{m+1}(X^m(\alpha, \tau), \tau)| d\tau. \quad (4.25)
\end{aligned}$$

With the help of Lemma 3.1, we get

$$\begin{aligned}
& \|\dot{\Delta}_j u^{m+1}(t)\|_{M_q^p} \\
& \leq \|\dot{\Delta}_j u^{m+1}(0)\|_{M_q^p} + \int_0^t \|[v^m \cdot \nabla, \dot{\Delta}_j] u^{m+1}(\tau)\|_{M_q^p} d\tau \\
& \quad + \int_0^t \|\dot{\Delta}_j((u^m \cdot \nabla)v^m)(\tau)\|_{M_q^p} d\tau + \int_0^t \|\dot{\Delta}_j \nabla \Pi^{m+1}(\tau)\|_{M_q^p} d\tau. \quad (4.26)
\end{aligned}$$

Multiplying both sides by  $2^{j(s-1)}$  and taking the  $\ell^r$  norm, it holds that

$$\begin{aligned}
& \|u^{m+1}(t)\|_{\dot{N}_{p,q,r}^{s-1}} \\
& \leq \|u^{m+1}(0)\|_{\dot{N}_{p,q,r}^{s-1}} + \int_0^t \left\| 2^{j(s-1)} \|[v^m \cdot \nabla, \dot{\Delta}_j] u^{m+1}(\tau)\|_{M_q^p} \right\|_{\ell^r} d\tau \\
& \quad + \int_0^t \|(u^m \cdot \nabla)v^m(\tau)\|_{\dot{N}_{p,q,r}^{s-1}} d\tau + \int_0^t \|\nabla \Pi^{m+1}(\tau)\|_{\dot{N}_{p,q,r}^{s-1}} d\tau \\
& =: I + II + III + IV. \quad (4.27)
\end{aligned}$$

From Bernstein's inequality, we have

$$I = \|\dot{\Delta}_m v_0(x)\|_{\dot{N}_{p,q,r}^{s-1}} \leq C 2^{-m} \|\dot{\Delta}_m v_0(x)\|_{\dot{N}_{p,q,r}^s} \leq C 2^{-m} \|v_0(x)\|_{\dot{N}_{p,q,r}^s}. \quad (4.28)$$

For the estimate of  $II$ , we have by Lemma 3.5 (taking  $p_1 = \infty$  and  $q_1 = q_2 = q$ )

$$\begin{aligned}
II & \leq C \int_0^t \left( \|\nabla v^m(\tau)\|_{L^\infty} \|u^{m+1}(\tau)\|_{\dot{N}_{p,q,r}^{s-1}} + \|u^{m+1}(\tau)\|_{L^\infty} \|v^m(\tau)\|_{\dot{N}_{p,q,r}^s} \right) d\tau \\
& \leq C \int_0^t \|v^m(\tau)\|_{\dot{N}_{p,q,r}^s} \|u^{m+1}(\tau)\|_{\dot{N}_{p,q,r}^{s-1}} d\tau. \quad (4.29)
\end{aligned}$$

For the estimate of  $III$ , it follows from Lemma 3.3 that

$$\begin{aligned}
III & \leq C \int_0^t \left( \|u^m(\tau)\|_{L^\infty} \|\nabla v^m(\tau)\|_{\dot{N}_{p,q,r}^{s-1}} + \|\nabla v^m(\tau)\|_{L^\infty} \|v^m(\tau)\|_{\dot{N}_{p,q,r}^{s-1}} \right) d\tau \\
& \leq C \int_0^t \|u^m(\tau)\|_{\dot{N}_{p,q,r}^{s-1}} \|v^m(\tau)\|_{\dot{N}_{p,q,r}^s} d\tau. \quad (4.30)
\end{aligned}$$

We can estimate  $\nabla \Pi^{m+1}$  as follows. From (4.23), it follows that

$$-\Delta \Pi^{m+1} = \operatorname{div}(v^m \cdot \nabla) u^{m+1} + \operatorname{div}(u^m \cdot \nabla) v^m, \quad (4.31)$$

which implies that

$$\partial_i \partial_j \Pi^{m+1} = R_i R_j \operatorname{div}(v^m \cdot \nabla) u^{m+1} + R_i R_j \operatorname{div}(u^m \cdot \nabla) v^m. \quad (4.32)$$

Thanks to  $\operatorname{div} v^m = 0$ , we have

$$\operatorname{div}(v^m \cdot \nabla) u^{m+1} = \sum_{k,l=1}^n \partial_k v_l^m \partial_l u_k^{m+1} = \sum_{k,l=1}^n \partial_l (\partial_k v_l^m u_k^{m+1}). \quad (4.33)$$

Hence, by Bernstein's inequality, it holds that

$$\begin{aligned} \|\nabla \Pi^{m+1}\|_{\dot{N}_{p,q,r}^{s-1}} &\leq C \sum_{i,j=1}^n \|\partial_i \partial_j \Pi^{m+1}\|_{\dot{N}_{p,q,r}^{s-2}} \\ &\leq C \sum_{k,l=1}^n \|R_i R_j \partial_k v_l^m v_k^{m+1}\|_{\dot{N}_{p,q,r}^{s-1}} + \|(u^m \cdot \nabla) v^m\|_{\dot{N}_{p,q,r}^{s-1}} \\ &\leq C \left( \|\nabla v^m\|_{L^\infty} \|u^{m+1}\|_{\dot{N}_{p,q,r}^{s-1}} + \|u^{m+1}\|_{L^\infty} \|\nabla v^m\|_{\dot{N}_{p,q,r}^{s-1}} \right) \\ &\quad + C \left( \|u^m\|_{L^\infty} \|\nabla v^m\|_{\dot{N}_{p,q,r}^{s-1}} + \|\nabla v^m\|_{L^\infty} \|u^m\|_{\dot{N}_{p,q,r}^{s-1}} \right) \\ &\leq C \|v^m\|_{\dot{N}_{p,q,r}^s} \|u^{m+1}\|_{\dot{N}_{p,q,r}^{s-1}} + \|v^m\|_{\dot{N}_{p,q,r}^s} \|u^m\|_{\dot{N}_{p,q,r}^{s-1}}, \end{aligned} \quad (4.34)$$

where we have taken  $p_1 = p_3 = \infty$ ,  $p_2 = p_4 = p$  and  $q_1 = q_3 = \infty$ ,  $q_2 = q_4 = q$  in Lemma 3.3.

Furthermore, we have

$$IV \leq C \int_0^t \|v^m(\tau)\|_{\dot{N}_{p,q,r}^s} \|u^{m+1}(\tau)\|_{\dot{N}_{p,q,r}^{s-1}} d\tau + C \int_0^t \|v^m(\tau)\|_{\dot{N}_{p,q,r}^s} \|u^m(\tau)\|_{\dot{N}_{p,q,r}^{s-1}} d\tau. \quad (4.35)$$

Taking the summation of (4.27)-(4.30) and (4.35), we conclude that

$$\begin{aligned} &\|u^{m+1}(t)\|_{\dot{N}_{p,q,r}^{s-1}} \\ &\leq C 2^{-m} \|v_0(x)\|_{N_{p,q,r}^s} + C \int_0^t \|v^m(\tau)\|_{\dot{N}_{p,q,r}^s} \|u^{m+1}(\tau)\|_{\dot{N}_{p,q,r}^{s-1}} d\tau \\ &\quad + C \int_0^t \|v^m(\tau)\|_{\dot{N}_{p,q,r}^s} \|u^m(\tau)\|_{\dot{N}_{p,q,r}^{s-1}} d\tau. \end{aligned} \quad (4.36)$$

Following from the similar procedure of estimate leading to (4.16), we get

$$\begin{aligned} \|u^{m+1}(t)\|_{M_q^p} &\leq C \|u^{m+1}(0)\|_{M_q^p} + C \int_0^t \|(u^m \cdot \nabla) v^m(\tau)\|_{M_q^p} d\tau \\ &\quad + C \int_0^t \|\nabla \Pi^{m+1}(\tau)\|_{M_q^p} d\tau, \end{aligned} \quad (4.37)$$

where the terms in the right side of (4.37) can be estimated as

$$\|u^{m+1}(0)\|_{M_q^p} = \|\dot{\Delta}_m v_0\|_{M_q^p} \leq C 2^{-m} \|\nabla v_0\|_{M_q^p} \leq C 2^{-m} \|v_0\|_{N_{p,q,r}^s}, \quad (4.38)$$

$$\begin{aligned}
\int_0^t \|(u^m \cdot \nabla)v^m(\tau)\|_{M_q^p} d\tau &\leq C \int_0^t \|\nabla v^m(\tau)\|_{L^\infty} \|u^m(\tau)\|_{M_q^p} d\tau \\
&\leq C \int_0^t \|v^m(\tau)\|_{N_{p,q,r}^s} \|u^m(\tau)\|_{N_{p,q,r}^{s-1}} d\tau
\end{aligned} \tag{4.39}$$

and

$$\begin{aligned}
\|\nabla \Pi^{m+1}\|_{M_q^p} &\leq \sum_{k=1}^n \|\nabla(-\Delta^{-1})\partial_k((u^{m+1} \cdot \nabla)v_k^m)\|_{M_q^p} \\
&\quad + \sum_{k=1}^n \|\nabla(-\Delta^{-1})\partial_k((u^m \cdot \nabla)v_k^m)\|_{M_q^p} \\
&\leq C\|(u^{m+1} \cdot \nabla)v^m\|_{M_q^p} + C\|(u^m \cdot \nabla)v^m\|_{M_q^p} \\
&\leq C\|\nabla v^m\|_{L^\infty} (\|u^{m+1}\|_{M_q^p} + \|u^m\|_{M_q^p}) \\
&\leq C\|v^m\|_{N_{p,q,r}^s} (\|u^{m+1}\|_{N_{p,q,r}^{s-1}} + \|u^m\|_{N_{p,q,r}^{s-1}}).
\end{aligned} \tag{4.40}$$

Therefore, from (4.37)-(4.40), we deduce that

$$\begin{aligned}
&\|u^{m+1}(t)\|_{M_q^p} \\
&\leq C2^{-m}\|v_0\|_{N_{p,q,r}^s} + C \int_0^t \|v^m(\tau)\|_{N_{p,q,r}^s} (\|u^{m+1}(\tau)\|_{N_{p,q,r}^{s-1}} + \|u^m(\tau)\|_{N_{p,q,r}^{s-1}}) d\tau.
\end{aligned} \tag{4.41}$$

Combining (4.36) and (4.41) gives

$$\begin{aligned}
&\|u^{m+1}(t)\|_{N_{p,q,r}^{s-1}} \\
&\leq C2^{-m}\|v_0\|_{N_{p,q,r}^s} + C \int_0^t \|v^m(\tau)\|_{N_{p,q,r}^s} (\|u^{m+1}(\tau)\|_{N_{p,q,r}^{s-1}} + \|u^m(\tau)\|_{N_{p,q,r}^{s-1}}) d\tau
\end{aligned} \tag{4.42}$$

for  $t \in [0, T]$ , which yields

$$\begin{aligned}
&\|u^{m+1}(t)\|_{N_{p,q,r}^{s-1}} \\
&\leq CC_1 2^{-m-1} + CC_1 T \|u^{m+1}\|_{L_{T_1}^\infty(N_{p,q,r}^{s-1})} + CC_1 T \|u^m\|_{L_{T_1}^\infty(N_{p,q,r}^{s-1})},
\end{aligned} \tag{4.43}$$

where  $C_1$  is the constant obtained for the uniform estimate. Furthermore, if we choose  $T_1 > 0$  sufficiently small so that  $CC_1 T_1 \leq 1/4$ , then

$$\begin{aligned}
&\|u^{m+1}\|_{L_{T_1}^\infty(N_{p,q,r}^{s-1})} \\
&\leq CC_1 2^{-m-1} + \frac{1}{2} \|u^{m+1}\|_{L_{T_1}^\infty(N_{p,q,r}^{s-1})} + \frac{1}{4} \|u^m\|_{L_{T_1}^\infty(N_{p,q,r}^{s-1})},
\end{aligned} \tag{4.44}$$

which leads to

$$\|u^{m+1}\|_{L_{T_1}^\infty(N_{p,q,r}^{s-1})} \leq \frac{CC_1}{2^m}, \quad m = 0, 1, 2, \dots \tag{4.45}$$

Due to (4.45), it is clear that  $\|u^{m+1}\|_{L_{T_1}^\infty(N_{p,q,r}^{s-1})} \rightarrow 0$ , as  $m$  tends to infinity. Therefore, there exists a limit  $v \in C([0, T_1]; N_{p,q,r}^{s-1})$  such that  $v^m(t) \rightarrow v(t)$  uniformly for  $t \in [0, T_1]$  in  $N_{p,q,r}^{s-1}$ .

Moreover, it is easy to see that  $v$  is a solution of (1.1). Indeed,  $v \in C([0, T_1]; N_{p,q,r}^s)$ . This completes the proof of the local existence part.

Step 4: Uniqueness

Suppose that  $v_1$  and  $v_2$  are two solutions of (1.1) with the same initial data. Set

$$\delta v = v_1 - v_2.$$

Then we get

$$\begin{cases} \partial_t \delta v + (v_1 \cdot \nabla) \delta v + (\delta v \cdot \nabla) v_2 + \nabla \tilde{\Pi} = 0, \\ \operatorname{div} v_1 = 0, \quad \operatorname{div} v_2 = 0, \\ v(x, 0) = 0 \end{cases} \quad (4.46)$$

where  $\tilde{\Pi} = P_1 - P_2$  with the associated pressures with  $v_1$  and  $v_2$ , respectively. We follow the strategy to derive the inequality (4.42) to obtain

$$\begin{aligned} & \|\delta v\|_{L_{T_1}^\infty(N_{p,q,r}^{s-1})} \\ & \leq CT_1 \|v_1\|_{L_{T_1}^\infty(N_{p,q,r}^s)} \|\delta v\|_{L_{T_1}^\infty(N_{p,q,r}^{s-1})} + CT_1 \|v_2\|_{L_{T_1}^\infty(N_{p,q,r}^s)} \|\delta v\|_{L_{T_1}^\infty(N_{p,q,r}^{s-1})} \\ & \leq 2CC_1 T_1 \|\delta v\|_{L_{T_1}^\infty(N_{p,q,r}^{s-1})}, \end{aligned} \quad (4.47)$$

where  $C_1 > 0$  is the constant obtained by the existence part. So if we choose  $T_1 > 0$  such that  $CC_1 T_1 \leq 1/4$ , then

$$\|\delta v\|_{L_{T_1}^\infty(N_{p,q,r}^{s-1})} \leq \frac{1}{2} \|\delta v\|_{L_{T_1}^\infty(N_{p,q,r}^{s-1})}, \quad (4.48)$$

which implies  $\delta v = 0$  for any  $t \in T_1$ , i.e.,  $v_1 \equiv v_2$  for any  $t \in T_1$ .

Step 5: Blow-up criterion

Suppose that  $v$  is the solution of (1.1) in the class  $C([0, T]; N_{p,q,r}^s)$ . As shown by [15], for the divergence free of  $v$ , we have the relation between the gradient of velocity and vorticity

$$\nabla v = \mathcal{P}(\omega) + A\omega, \quad (4.49)$$

where  $\mathcal{P}$  is a singular integral operator homogeneous of degree  $-n$  and  $A$  is a constant matrix. By the boundedness of the singular integral operator from  $\dot{B}_{\infty,\infty}^0$  into itself [19], and Lemma 3.2, we get

$$\begin{aligned} \|\nabla v\|_{L^\infty} & \leq C \left( 1 + \|\nabla v\|_{\dot{B}_{\infty,\infty}^0} (\log^+ \|\nabla v\|_{N_{p,q,r}^{s-1}} + 1) \right) \\ & \leq C \left( 1 + \|\omega\|_{\dot{B}_{\infty,\infty}^0} (\log^+ \|v\|_{N_{p,q,r}^s} + 1) \right) \end{aligned} \quad (4.50)$$

for  $s > 1 + n/p$ .

As (4.19) previously, we obtain similarly

$$\|v(t)\|_{N_{p,q,r}^s} \leq C \|v_0\|_{N_{p,q,r}^s} + C \int_0^t \|\nabla v(\tau)\|_{L^\infty} \|v(\tau)\|_{N_{p,q,r}^s} d\tau. \quad (4.51)$$

Substituting (4.50) into (4.51) to get

$$\begin{aligned} & \|v(t)\|_{N_{p,q,r}^s} \\ & \leq C\|v_0\|_{N_{p,q,r}^s} + C \int_0^t \left(1 + \|\omega(\tau)\|_{\dot{B}_{\infty,\infty}^0} (\log^+ \|v(\tau)\|_{N_{p,q,r}^s} + 1)\right) \|v(\tau)\|_{N_{p,q,r}^s} d\tau, \end{aligned} \quad (4.52)$$

which implies that

$$\|v(t)\|_{N_{p,q,r}^s} \leq C_2\|v_0\|_{N_{p,q,r}^s} \exp \left[ C_3 \exp \left( C_4 \int_0^t (1 + \|\omega(\tau)\|_{\dot{B}_{\infty,\infty}^0}) d\tau \right) \right], \quad (4.53)$$

by Gronwall's inequality. Here  $C_2, C_3$  and  $C_4$  are some positive constants. Therefore, if  $\limsup_{t \rightarrow T^* -} \|v(t)\|_{N_{p,q,r}^s} = \infty$ , then  $\int_0^{T^*} \|\omega(t)\|_{\dot{B}_{\infty,\infty}^0} dt = \infty$ .

On the other hand, it follows from Sobolev embedding  $N_{p,q,r}^s \hookrightarrow L^\infty \hookrightarrow \dot{B}_{\infty,\infty}^0$  for  $s > 1 + n/p$  that

$$\int_0^{T^*} \|\omega(t)\|_{\dot{B}_{\infty,\infty}^0} dt \leq \int_0^{T^*} \|\nabla v(t)\|_{L^\infty} dt \leq T^* \sup_{t \in [0, T^*]} \|v(t)\|_{N_{p,q,r}^s}. \quad (4.54)$$

Then  $\int_0^{T^*} \|\omega(t)\|_{\dot{B}_{\infty,\infty}^0} dt = \infty$  implies that  $\limsup_{t \rightarrow T^* -} \|v(t)\|_{N_{p,q,r}^s} = \infty$ .

Besides, for  $s = 1 + n/p$ , since  $\dot{B}_{\infty,1}^0 \hookrightarrow L^\infty$  and the singular integral operator  $\mathcal{P}$  is bounded from  $\dot{B}_{\infty,1}^0$  into itself, we have

$$\|\nabla v\|_{L^\infty} \leq C\|\nabla v\|_{\dot{B}_{\infty,1}^0} \leq C\|\omega\|_{\dot{B}_{\infty,1}^0}. \quad (4.55)$$

Substituting (4.55) into (4.51), we have

$$\|v(t)\|_{N_{p,q,r}^s} \leq C\|v_0\|_{N_{p,q,r}^s} + C \int_0^t \|\omega(\tau)\|_{\dot{B}_{\infty,1}^0} \|v(\tau)\|_{N_{p,q,r}^s} d\tau. \quad (4.56)$$

Then Gronwall's inequality gives

$$\|v(t)\|_{N_{p,q,r}^s} \leq C_5\|v_0\|_{N_{p,q,r}^s} \exp \left( C_6 \int_0^t \|\omega(\tau)\|_{\dot{B}_{\infty,1}^0} d\tau \right) \quad (4.57)$$

for some positive constants  $C_5$  and  $C_6$ .

On the other hand, it follows from the Sobolev embedding  $N_{p,q,1}^{n/p} \hookrightarrow \dot{N}_{p,q,1}^{n/p} \hookrightarrow \dot{B}_{\infty,1}^0$  that

$$\int_0^T \|\omega(t)\|_{\dot{B}_{\infty,1}^0} dt \leq \int_0^T \|\nabla v(t)\|_{N_{p,q,1}^{n/p}} dt \leq T \sup_{t \in [0, T]} \|v(t)\|_{N_{p,q,1}^{1+n/p}}. \quad (4.58)$$

(4.57)-(4.58) implies the blow-up criterion for the case of  $s = 1 + n/p$ .

#### Step 6: Solve the linear equations

To finish the Proof of Theorem 1.1, what left is to solve the linear equations (4.1). Our idea is to approximate (4.1) by the linear transport equations. First, we see that (4.1) is equivalent to the following system

$$\begin{cases} \partial_t v + (w \cdot \nabla) v + \nabla P = 0, \\ -\Delta P = \operatorname{div}((w \cdot \nabla) v), \\ v(x, 0) = v_0(x), \operatorname{div} v_0 = 0, \end{cases} \quad (4.59)$$

which can be approximated by the following linear transport equations

$$\begin{cases} \partial_t v^{n+1} + (w \cdot \nabla) v^n + \nabla P^n = 0, \\ -\Delta P^n = \operatorname{div}((w \cdot \nabla) v^n), \\ v^{n+1}(x, 0) = S_{n+1} v_0. \end{cases} \quad (4.60)$$

The existence theorem for (4.60) is well-known for each  $n$ . To prove the solvability of (4.59), it is suffice to prove the uniform estimate for the sequence  $\{v^{n+1}\}$  in the  $N_{p,q,r}^s$  framework and the Cauchy convergence of the corresponding sequence. Indeed, This depends on the following a priori estimates for (4.60), which can be shown in a similar manner with (4.14) and (4.18). Precisely,

$$\begin{aligned} \|v(t)\|_{\dot{N}_{p,q,r}^s} &\leq C \|v_0\|_{\dot{N}_{p,q,r}^s} + C \int_0^t \|\nabla P(\tau)\|_{\dot{N}_{p,q,r}^s} d\tau + C \int_0^t \|2^{js} \|[w \cdot \nabla, \dot{\Delta}_j]v(\tau)\|_{M_q^p} d\tau \\ &\leq C \|v_0\|_{\dot{N}_{p,q,r}^s} + C \int_0^t \|w\|_{\dot{N}_{p,q,r}^s} \|v\|_{\dot{N}_{p,q,r}^s} d\tau \end{aligned} \quad (4.61)$$

and

$$\begin{aligned} \|v(t)\|_{M_q^p} &\leq C \|v_0\|_{M_q^p} + C \int_0^t \|\nabla P(\tau)\|_{M_q^p} d\tau \\ &\leq C \|v_0\|_{M_q^p} + C \int_0^t \|w\|_{N_{p,q,r}^s} \|v\|_{\dot{N}_{p,q,r}^s} d\tau. \end{aligned} \quad (4.62)$$

Hence, we easily arrive at

$$\|v(t)\|_{N_{p,q,r}^s} \leq C \|v_0\|_{N_{p,q,r}^s} + C \int_0^t \|w\|_{N_{p,q,r}^s} \|v\|_{N_{p,q,r}^s} d\tau. \quad (4.63)$$

Applying Gronwall's inequality on (4.63) to get

$$\|v(t)\|_{N_{p,q,r}^s} \leq C \|v_0\|_{N_{p,q,r}^s} \exp \left( C \int_0^t \|w(t)\|_{N_{p,q,r}^s} dt \right), \quad t \in [0, T]. \quad (4.64)$$

Having the a priori estimate (4.64), the existence and uniqueness of solutions for the system (4.59) can be obtained by the approximate solution sequence  $\{v^{n+1}\}$  of (4.60). This finished the proof of Proposition 4.1.

Above all, we complete the proof of Theorem 1.1 eventually.

## 5 Proof of Theorem 1.2

In the similar spirit, we can prove the Theorem 1.2. In comparison with the Euler equations (1.1), it is suffice to handle with the coupling between the velocity field and magnetic field in the MHD system (1.2). Therefore, we only give the crucial estimates for conciseness. First, we consider the linear equations of MHD system:

$$\begin{cases} \partial_t v + (w \cdot \nabla) v - (a \cdot \nabla) b + \nabla \Pi = 0, \\ \partial_t b + (w \cdot \nabla) b - (a \cdot \nabla) v = 0, \\ \operatorname{div} v = 0, \quad \operatorname{div} b = 0, \\ v(x, 0) = v_0(x), \quad b(x, 0) = b_0. \end{cases} \quad (5.1)$$

For (5.1), similar to the Proposition 4.1, we have

**Proposition 5.1.** *Assume that  $\operatorname{div} w = \operatorname{div} a = 0$ ,  $(w, a) \in L^\infty(0, T, N_{p,q,r}^s)$  for some  $T > 0$ ,  $s > 1 + n/p$ ,  $1 < q \leq p < \infty$ ,  $r \in [1, \infty]$  or  $s = 1 + n/p$ ,  $1 < q \leq p < \infty$  and  $r = 1$ . Then for any  $(v_0, b_0) \in N_{p,q,r}^s$  and  $\operatorname{div} v_0 = \operatorname{div} b_0 = 0$ , there exists a unique solution  $(v, b) \in C([0, T]; N_{p,q,r}^s)$  to the linear system (5.1). And consequently,  $\nabla \Pi$  can be uniquely determined.*

Based on Proposition 5.1, we construct the following approximate linear system of (1.2)

$$\begin{cases} \partial_t v^{m+1} + (v^m \cdot \nabla) v^{m+1} - (b^m \cdot \nabla) b^{m+1} + \nabla \Pi^{m+1} = 0, \\ \partial_t b^{m+1} + (v^m \cdot \nabla) b^{m+1} - (b^m \cdot \nabla) v^{m+1} = 0, \\ \operatorname{div} v^{m+1} = \operatorname{div} v^m = 0, \quad \operatorname{div} b^{m+1} = \operatorname{div} b^m = 0, \\ v_0^{m+1} = S_{m+1} v_0(x), \quad b_0^{m+1} = S_{m+1} b_0. \end{cases} \quad (5.2)$$

for  $m = 0, 1, 2, \dots$ , where we set  $v^0 = b^0 = 0$ .

In what follows, we give the uniform estimates for approximate solution sequence  $\{(v^{m+1}, b^{m+1})\}$ . Indeed, we perform  $\dot{\Delta}_j (j \in \mathbb{Z})$  on the first two equations of (5.2) to get

$$\begin{cases} \partial_t \dot{\Delta}_j v^{m+1} + (v^m \cdot \nabla) \dot{\Delta}_j v^{m+1} - (b^m \cdot \nabla) \dot{\Delta}_j b^{m+1} \\ = [v^m \cdot \nabla, \dot{\Delta}_j] v^{m+1} - [b^m \cdot \nabla, \dot{\Delta}_j] b^{m+1} - \dot{\Delta}_j \nabla \Pi^{m+1}, \\ \partial_t \dot{\Delta}_j b^{m+1} + (v^m \cdot \nabla) \dot{\Delta}_j b^{m+1} - (b^m \cdot \nabla) \dot{\Delta}_j v^{m+1} \\ = [v^m \cdot \nabla, \dot{\Delta}_j] b^{m+1} - [b^m \cdot \nabla, \dot{\Delta}_j] v^{m+1}. \end{cases} \quad (5.3)$$

To deal with the coupling of  $v^{m+1}$  and  $b^{m+1}$ , similar to the particle trajectory mapping  $\{X^m(\alpha, t)\}$ , we define  $\{Y^m(\alpha, t)\}$  as follows

$$\begin{cases} \partial_t Y^m(\alpha, t) = (v^m - b^m)(Y^m(\alpha, t), t), \\ Y^m(\alpha, 0) = \alpha. \end{cases} \quad (5.4)$$

Note that  $\operatorname{div}(v^m - b^m) = 0$  implies that each  $\alpha \mapsto Y^m(\alpha, t)$  is a volume-preserving mapping for all  $t > 0$ . So it follows from the particle trajectory mapping (5.4) that

$$\begin{aligned} \partial_t \dot{\Delta}_j (v^{m+1} + b^{m+1}) + [(v^m - b^m) \cdot \nabla] \dot{\Delta}_j (v^{m+1} + b^{m+1}) \Big|_{(x,t)=(Y^m(\alpha,t),t)} \\ = \frac{\partial}{\partial t} \dot{\Delta}_j (v^{m+1} + b^{m+1})(Y^m(\alpha, t), t) \end{aligned} \quad (5.5)$$

which yields

$$\begin{aligned} & |\dot{\Delta}_j (v^{m+1} + b^{m+1})(Y^m(\alpha, t), t)| \\ \leq & |\dot{\Delta}_j (v^{m+1} + b^{m+1})(\alpha, 0)| + \int_0^t |\dot{\Delta}_j \nabla \Pi^{m+1}(Y^m(\alpha, \tau), \tau)| d\tau \\ & + \int_0^t |[v^m \cdot \nabla, \dot{\Delta}_j] v^{m+1}(Y^m(\alpha, \tau), \tau)| d\tau + \int_0^t |[b^m \cdot \nabla, \dot{\Delta}_j] b^{m+1}(Y^m(\alpha, \tau), \tau)| d\tau \\ & + \int_0^t |[v^m \cdot \nabla, \dot{\Delta}_j] b^{m+1}(Y^m(\alpha, \tau), \tau)| d\tau + \int_0^t |[b^m \cdot \nabla, \dot{\Delta}_j] v^{m+1}(Y^m(\alpha, \tau), \tau)| d\tau. \end{aligned} \quad (5.6)$$

As (4.14), we deduce similarly

$$\begin{aligned}
& \|v^{m+1}(t) + b^{m+1}(t)\|_{\dot{N}_{p,q,r}^s} \\
& \leq \|v_0^{m+1}\|_{\dot{N}_{p,q,r}^s} + \|b_0^{m+1}\|_{\dot{N}_{p,q,r}^s} + C \int_0^t \|\nabla \Pi^{m+1}(\tau)\|_{\dot{N}_{p,q,r}^s} d\tau \\
& \quad + C \int_0^t \left\| 2^{js} \|[v^m \cdot \nabla, \dot{\Delta}_j]v^{m+1}(\tau)\|_{M_q^p} \right\|_{\ell^r} d\tau + C \int_0^t \left\| 2^{js} \|[b^m \cdot \nabla, \dot{\Delta}_j]b^{m+1}(\tau)\|_{M_q^p} \right\|_{\ell^r} d\tau \\
& \quad + C \int_0^t \left\| 2^{js} \|[v^m \cdot \nabla, \dot{\Delta}_j]b^{m+1}\|_{M_q^p} \right\|_{\ell^r} d\tau + C \int_0^t \left\| 2^{js} \|[b^m \cdot \nabla, \dot{\Delta}_j]v^{m+1}\|_{M_q^p} \right\|_{\ell^r} d\tau, \\
& \leq \|v_0^{m+1}\|_{\dot{N}_{p,q,r}^s} + \|b_0^{m+1}\|_{\dot{N}_{p,q,r}^s} + C \int_0^t \|\nabla \Pi^{m+1}(\tau)\|_{\dot{N}_{p,q,r}^s} d\tau \\
& \quad + C \int_0^t (\|v^m(\tau)\|_{\dot{N}_{p,q,r}^s} + \|b^m(\tau)\|_{\dot{N}_{p,q,r}^s})(\|v^{m+1}(\tau)\|_{\dot{N}_{p,q,r}^s} + \|b^{m+1}(\tau)\|_{\dot{N}_{p,q,r}^s}) d\tau, \tag{5.7}
\end{aligned}$$

where we have used the commutator estimate in Lemma 3.4.

From (5.2), it follows that  $\Delta \Pi^{m+1} = \operatorname{div}((v^m \cdot \nabla)v^{m+1} - (b^m \cdot \nabla)b^{m+1})$ , which implies

$$\partial_i \partial_j \Pi^{m+1} = -R_i R_j \operatorname{div}((v^m \cdot \nabla)v^{m+1} - (b^m \cdot \nabla)b^{m+1}). \tag{5.8}$$

Since  $\operatorname{div} v^m = \operatorname{div} b^m = 0$ , we have

$$\operatorname{div}(v^m \cdot \nabla)v^{m+1} = \sum_{k,l=1}^n \partial_k v_l^m \partial_l v_k^{m+1} = \sum_{k,l=1}^n \partial_l (\partial_k v_l^m v_k^{m+1}) \tag{5.9}$$

and

$$\operatorname{div}(b^m \cdot \nabla)b^{m+1} = \sum_{k,l=1}^n \partial_k b_l^m \partial_l b_k^{m+1} = \sum_{k,l=1}^n \partial_l (\partial_k b_l^m b_k^{m+1}). \tag{5.10}$$

Hence, by Bernstein's inequality, we have

$$\begin{aligned}
\|\nabla \Pi^{m+1}\|_{\dot{N}_{p,q,r}^s} & \leq C \sum_{i,j=1}^n \|\partial_i \partial_j \Pi^{m+1}\|_{\dot{N}_{p,q,r}^{s-1}} \\
& \leq C \|v^m\|_{\dot{N}_{p,q,r}^s} \|v^{m+1}\|_{\dot{N}_{p,q,r}^s} + \|b^m\|_{\dot{N}_{p,q,r}^s} \|b^{m+1}\|_{\dot{N}_{p,q,r}^s}. \tag{5.11}
\end{aligned}$$

Substitute (5.11) into (5.7) to get

$$\begin{aligned}
& \|v^{m+1}(t) + b^{m+1}(t)\|_{\dot{N}_{p,q,r}^s} \\
& \leq \|v_0^{m+1}\|_{\dot{N}_{p,q,r}^s} + \|b_0^{m+1}\|_{\dot{N}_{p,q,r}^s} + C \int_0^t (\|v^m(\tau)\|_{\dot{N}_{p,q,r}^s} + \|b^m(\tau)\|_{\dot{N}_{p,q,r}^s}) \\
& \quad \times (\|v^{m+1}(\tau)\|_{\dot{N}_{p,q,r}^s} + \|b^{m+1}(\tau)\|_{\dot{N}_{p,q,r}^s}) d\tau. \tag{5.12}
\end{aligned}$$

Next, we turn to estimate the  $M_q^p$  norm. With the help of the particle trajectory mapping (5.4), we obtain

$$|(v^{m+1} + b^{m+1})(X^m(\alpha, t), t)| \leq |(v^{m+1} + b^{m+1})(\alpha, 0)| + \int_0^t |\nabla \Pi^{m+1}(Y^m(\alpha, \tau), \tau)| d\tau. \tag{5.13}$$



Furthermore, it follows from the fact  $\det \nabla_\alpha Y^m(\alpha, t) \equiv 1$  that

$$\begin{aligned}
& \| (v^{m+1} + b^{m+1})(t) \|_{M_q^p} \\
& \leq \| v_0^{m+1} + b_0^{m+1} \|_{M_q^p} + C \int_0^t \left( \| v^m \|_{M_q^p} \| \nabla v^{m+1} \|_{L^\infty} + \| b^m \|_{M_q^p} \| \nabla b^{m+1} \|_{L^\infty} \right) d\tau \\
& \leq \| v_0^{m+1} \|_{M_q^p} + \| b_0^{m+1} \|_{M_q^p} \\
& \quad + C \int_0^t \left( \| v^m(\tau) \|_{N_{p,q,r}^s} \| v^{m+1}(\tau) \|_{N_{p,q,r}^s} + \| b^m(\tau) \|_{N_{p,q,r}^s} \| b^{m+1}(\tau) \|_{N_{p,q,r}^s} \right) d\tau, \quad (5.14)
\end{aligned}$$

where we have used the Lemma 2.5.

Adding (5.14) to (5.12) together, we arrive at

$$\begin{aligned}
& \| v^{m+1}(t) + b^{m+1}(t) \|_{N_{p,q,r}^s} \\
& \leq \| v_0 \|_{N_{p,q,r}^s} + \| b_0 \|_{N_{p,q,r}^s} + C \int_0^t (\| v^m(\tau) \|_{N_{p,q,r}^s} + \| b^m(\tau) \|_{N_{p,q,r}^s}) \\
& \quad \times (\| v^{m+1}(\tau) \|_{N_{p,q,r}^s} + \| b^{m+1}(\tau) \|_{N_{p,q,r}^s}) d\tau. \quad (5.15)
\end{aligned}$$

Besides, We define by  $\{Z^m(\alpha, t)\}$  the family of particle trajectory mapping

$$\begin{cases} \partial_t Z^m(\alpha, t) = (v^m + b^m)(Z^m(\alpha, t), t), \\ Z^m(\alpha, 0) = \alpha. \end{cases} \quad (5.16)$$

Note that  $\operatorname{div}(v^m + b^m) = 0$  implies that each  $\alpha \mapsto Z^m(\alpha, t)$  is a volume-preserving mapping for all  $t > 0$ . It follows from the particle trajectory mapping (5.16) that

$$\begin{aligned}
& \partial_t \dot{\Delta}_j(v^{m+1} - b^{m+1}) + [(v^m + b^m) \cdot \nabla] \dot{\Delta}_j(v^{m+1} - b^{m+1}) \Big|_{(x,t)=(Z^m(\alpha,t),t)} \\
& = \frac{\partial}{\partial t} \dot{\Delta}_j(v^{m+1} - b^{m+1})(Z^m(\alpha, t), t). \quad (5.17)
\end{aligned}$$

Similar to (5.15), we can deduce that

$$\begin{aligned}
& \| v^{m+1}(t) - b^{m+1}(t) \|_{N_{p,q,r}^s} \\
& \leq \| v_0 \|_{N_{p,q,r}^s} + \| b_0 \|_{N_{p,q,r}^s} + C \int_0^t (\| v^m(\tau) \|_{N_{p,q,r}^s} + \| b^m(\tau) \|_{N_{p,q,r}^s}) \\
& \quad \times (\| v^{m+1}(\tau) \|_{N_{p,q,r}^s} + \| b^{m+1}(\tau) \|_{N_{p,q,r}^s}) d\tau. \quad (5.18)
\end{aligned}$$

Together with (5.15) and (5.18), we conclude that

$$\begin{aligned}
& \| v^{m+1}(t) \|_{N_{p,q,r}^s} + \| b^{m+1}(t) \|_{N_{p,q,r}^s} \\
& \leq C \| v_0 \|_{N_{p,q,r}^s} + C \| b_0 \|_{N_{p,q,r}^s} + C \int_0^t (\| v^m(\tau) \|_{N_{p,q,r}^s} + \| b^m(\tau) \|_{N_{p,q,r}^s}) \\
& \quad \times (\| v^{m+1}(\tau) \|_{N_{p,q,r}^s} + \| b^{m+1}(\tau) \|_{N_{p,q,r}^s}) d\tau. \quad (5.19)
\end{aligned}$$

By using the Gronwall's inequality, we get

$$\begin{aligned}
& \| v^{m+1}(t) \|_{N_{p,q,r}^s} + \| b^{m+1}(t) \|_{N_{p,q,r}^s} \\
& \leq C \left( \| v_0 \|_{N_{p,q,r}^s} + \| b_0 \|_{N_{p,q,r}^s} \right) \exp \left\{ C \int_0^t (\| v^m(\tau) \|_{N_{p,q,r}^s} + \| b^m(\tau) \|_{N_{p,q,r}^s}) d\tau \right\}. \quad (5.20)
\end{aligned}$$

Based on the above crucial estimates, we can finish the proof of Theorem 1.2 following from the subsequent steps of the proof of Theorem 1.1. We would like to skip the details, for conciseness.

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